Status Seminar
Dissipation and dispersion in nonlinear PDEs

ADAPTIVE ALGORITHMS FOR THE NUMERICAL TREATMENT OF PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. Given a partial differential equation and a numerical algorithm to solve it, the overall goal is to compute a satisfying approximation of the unknown exact solution while investing a minimal amount of computational power and time. Usually, singularities in the solution and in the data spoil the performance of numerical algorithms, since they require a fine resolution to get a decent approximation. In this context, adaptive algorithms are designed with the goal to distribute the available resources such that difficult parts of the problem, i.e. singularities, are resolved in great detail, whereas easy parts of the problem, i.e. smooth solution and data, are treated only superficially. To that end, it is essential to have an estimate of the distance between the approximation and the exact solution, without knowing the exact solution, namely an error estimator.

In this talk, we apply this general idea to variational problems of the form

$$b(u,v) = f(v) \quad \text{for all } v \in \mathcal{X}, \quad \text{e.g. } b(u,v) := \int_\Omega \nabla u \cdot \nabla v \, dx$$

for some Hilbert space $\mathcal{X}$ and some bilinear form $b(\cdot,\cdot)$ with corresponding right-hand side $f \in \mathcal{X}^\star$, which are solved numerically on a finite dimensional subspace

$$b(U_0,V) = f(V) \quad \text{for all } V \in \mathcal{X}_0 \subset \mathcal{X}, \quad \text{e.g. } \mathcal{X}_0 = \{\text{piecewise polynomials on } \Omega\}.$$

Moreover, we require that the problem allows for a computable error estimator of the form

$$\rho(U,\mathcal{X}_0) : \mathcal{X}_0 \to [0,\infty).$$

The strategy is, roughly speaking, to enrich the space $\mathcal{X}_0$ only where $\rho(U,\mathcal{X}_0)$ is big and to generate an improved space $\mathcal{X}_1 \supseteq \mathcal{X}_0$ as well as an improved approximation U_1. The iteration of this process raises natural questions like convergence

$$\lim_{\ell\to\infty} \|u - U_\ell\| = 0$$

and approximation quality, i.e. find minimal $\mathcal{X}_\ell$ such that $\|u - U_\ell\| \lesssim \epsilon$.